Time Value of Money

Jean-Charles Bagneris

v2017.09.1

Abstract

Basic principles and formulas of time value of money calculations.
Keywords: value, cash flow, present value, future value, discount rate, discounting, annuity

Contents

Learning objectives

By the end of this module, students should be able to:

- Explain the origin and principles of time value of money calculation
- Understand the definition and use of the discount rate
- Define simple and compound interests methods
- Choose and use appropriate formulas for various calculations such as discounting, calculating annuities
- Build a loan repayment table under three different repayment methods: interest only loan, constant repayments, constant annuities

1 Introduction: time and value

We are all aware of the time passing, even if we don't think of it routinely. One of the consequences of this notion of time is that we can think of, and anticipate, future events. Some of these (potential) events we might fear, and we might also long for some others. Kids want their birthday party to come as soon as possible, adults want to get their wages on time and see any delay as a terrible offense.

In finance, we generally suppose that agents fear uncertainty (they are *risk averse*) and that they prefer to receive money earlier than later (they experience *time preference*). These two factors, risk aversion and time preference, have an impact on the value of future cash flows (that is, money we expect to receive at a given future date): the later the cash flows are expected to be received[1], the lower their perceived value today. This impact of time on value is known as *time value of money* and can be precisely measured with the tools we introduce here.

The time value of money calculations should thus take into account two factors: the risk aversion and the time preference.

Time preference affects the calculations through a time variable, more precisely, a variable which measures how many periods of time (days, months, years…) we have to wait to receive (or pay) the cash flow. Of course, the greater the number of periods we have to wait, the lower the value we assess now to the expected cash flow.

Risk aversion is taken into account through a *risk premium* in time value of money calculations. Think of it that way: risk aversion does not mean you are not willing to take risk, only that the greater the risk, the greater the reward you expect. The risk premium is the additional reward you get for the risk level you bear. The determination of the risk premium goes far beyond the scope of this document[2], but we have to understand how to take it into account, and through which variable. Let me use an example to illustrate this.

> **Example 1**
>
> Imagine you just got hired for a summer job in a bank. Your job supposes you dress formally, so

[1] Note that expecting to receive or to pay a cash flow are symmetrical operations, only the direction changes: when someone receives money, someone pays it (a payment is a transfer). Thus, in the following, receiving or paying refer to the same operation, a money transfer. The only difference is the point of view, or, which agent or side we are considering.

[2] To learn about this topic, see the "Risk and return" chapter in any finance textbook.

> you need to buy some formal clothes. The problem is that those are costly, and until you get your first wages in the end of the month, you cannot afford this spending. In finance terms, you expect to receive a future cash flow in one month (period) but you need the money now.
>
> In our world, the solution is easy: credit, that is, borrow the money (remember you have been hired by a bank). Of course, you will have to pay for this service (lending the money to you for one period of time[a]). The payment in this case is known as interest, as you know, and depends both on an *interest rate*, and on the *duration* of the loan (one period here).
>
> The duration of the loan is the time variable, taking into account the time preference – you prefer (or need) to get the money now.
>
> Where is the risk aversion in this example? Well, by lending you money, the bank takes a risk: what if you cannot repay for the loan and pay interests in the end of the month? The bank, as any agent, is risk averse and will assess the risk it takes by lending this amount, for this period of time, to you (here the risk is low because the amount is low, the time is short, and you are an employee of the bank). The risk assessment will result in a risk premium that it will add to the basis interest rate it charges to its customer: as we have seen, they agree to take the risk, but take a higher reward because of it.
>
> Finally note that the amount you can spend now (theoretically your expected wages) is reduced by the amount of interests you will have to pay in the end of the month (because with your wages you will have to repay for the loan *and* pay the interests in addition).
>
> [a] The ethical and philosophical justification of the interest is far beyond the scope of these pages. Simply note that the concept of interest is not obvious and had been, and still is, disputed. Actually, some branches of finance (namely Islamic finance) exclude the notion of interest, and replace it with different ways to share profit and risk between lender and borrower.

So, in this example the risk premium is hidden in the interest rate. Actually, in time value of money calculations, the risk premium is always hidden in the rate, which is called *discount rate*. The discount rate might be an interest rate, but it is better defined as the opportunity cost of the resources.

Again, the higher the risk, the higher the premium and the rate, and finally the interests you have to pay: the higher the risk, the lower the value you assess now to the future cash flow (the lower the amount you could spend in the example).

The value you assess now to an expected future cash flow is called the cash flow's present value, and usually noted *PV*, or better, CF_0, to clearly show that it is the value at time $t = 0$ (now, present time). We can now restate what we just observed about the effects of time preference and risk aversion on value:

- the greater the number of periods before we expect to receive a cash flow, the lower its present value,
- the greater the risk associated with the expected cash flow, the lower its present value.

In the following, we will translate these statements in basic formulas, and then introduce useful transformations of the formulas for some real life cases, such as loan repayments.

2 Definitions and basis formulas

2.1 Cash flows, periods, and financial decisions

Many if not all financial operations can be described with a series of cash flows, each cash flow having a value, a direction or sign (paid or received) and an occurrence date.

Example 2

A 2,000 loan with a 10% interest rate will be repaid by two installments, one in one year and one in two years. From the borrower point of view, the series of cash flows is as follows (the interest calculations will be detailed in section 2.4 below):

Year	0	1	2
Cash flow	2,000	-1,200	-1,100

By convention, today is written *t*=0, as what is relevant is how far in the future a cash flow is from today, thus dates are computed by difference with today (*i.e.* they are relative).

Financial management involves taking decisions, and when one faces a decision, it should be taken usually as soon as possible, that is, “now”. To take the decision, we have to take into account the series of cash flows associated with the underlying financial operation, many of which are *future* cash flows. The decision often implies comparing cash flows, either inside the series or comparing two series of cash flows. To take a sane decision, we need to express all the cash flows at the same point in time (for valid comparisons), and, as we are supposed to decide now, this point in time is now, *t*=0. This is the reason why we calculate and use *present values* of cash flows in most financial decisions. Always remember that to combine (add) or compare (subtract) values, they should be expressed at the same point in time – and most of the time, the chosen point in time is now: present values are ubiquitous in most financial decision processes.

2.2 Present value and discounting

Calculating a present value is a bit like a time travel for money: we calculate how much of a future cash flow would be available today if we had to find a way to make the money available immediately.

As we have seen previously with example 1, it is actually possible to do it for real, make the future money available immediately. One of the ways of doing it is to get financing immediately, and incurs the cost of this financing. Another way would be to allocate some money from one project to another: we would then incur what is called a cost of opportunity, as we renounce the potential reward of the original project, for the new one. Either way, the future cash flow will then be used to compensate for the method we used: repay a loan, re-allocate money to the forsaken project.

You understand immediately the point: whatever the way we use to get money immediately from a future cash flow, there is an associated cost, be it cost of financing or opportunity cost.

You might object that we also could have immediately available resources, which would incur no cost at all. Then these resources could be saved on a savings account, for an example, producing interests. Using it for something else has then a cost: the renounced interests.

Calculating a present value is simply removing (subtracting) the financing or opportunity cost from a future cash flow. As we have seen already, this cost depends on time and grows with it, it also depends on a rate (interest rate, cost of financing, opportunity rate like the interest rate on savings etc.), and grows with the rate as well. This rate is known as the *discount rate*, as we use it to discount.

To discount a future cash flow is to calculate its present value. For this, we need the cash flow future value, the time at which the cash flow is supposed to be available, and a discount rate.

2.3 Future value and capitalizing

Some financing operations are meant to prepare for the future: savings, insurance, retirement preparation etc. In this case, we usually know how much we want to set aside to anticipate, and we want to calculate how much will be available at a future time.

This operation is exactly the reverse of discounting, and is called capitalizing. Instead of removing the "cost of time" from the future value, we add the "gain of time" to the present value to find the future value.

Here again, we need the cash flow present value, the time at which we want to get the future value and a rate (known as capitalization rate) to perform the calculation.

2.4 The one period case

Interest

As we have just seen, the discount rate is not always an interest rate, as it might be the cost of other sources of financing, or even an opportunity cost. It is nonetheless useful to focus on the interests case, without any loss of generality: the discounting and capitalizing formulas will be the same, whatever the precise nature of the discount or capitalization rate.

In the interests case, the cost of financing is the interest itself, that is, the cash flow payment which is due by the borrower in addition to the capital repayment to reward the lender. As discounting is equivalent to removing the cost of financing from a future cash flow, discounting in the interests case means removing the interests from the future cash flow. We thus have to calculate the interests.

Luckily, there is only one way to calculate the interests due over one period of time – complications arise when we have more than one period.

Over one period of time, the interest is *always* calculated on the unpaid balance (for a loan), or the amount of capital on the account (for savings), at the **beginning of the period**. This balance is simply multiplied by the interest rate for one period, known as *periodic rate*, to yield the amount of interests.

We get:

$$Int = UB \times r \tag{1}$$

Int interest over the period
r interest rate for one period (periodic rate)
UB unpaid balance in the beginning of the period

Example 3

A 1,200 savings on an account which pays 3% per year would yield after one year (assuming of course there is no movement on the savings account over the year):

$$1,200 \times 3\% = 1,200 \times 3/100 = 36$$

Again, the interest rate basis and the period should be consistent: in the example above, the interest year is given "per year", and the period is one year, so it's OK. When the basis is different, it is necessary to calculate a periodic rate.

Present and future values

Using the formula above, we can calculate that the lender on a one period loan will get in the end of the period[3] a total cash flow which is the sum of the capital repayment and the interests. That is, the *future value* of the capital lent in the beginning of the period is this capital plus the interests. In a symmetric way, the *present value* of the cash flow the lender will get in the end of the period is this cash flow, minus the interests (cost of financing for the borrower), that is, the initial capital. Let us write all this:

CF_0 present value
CF_1 future value at time $t = 1$
r interest rate for one period
Int interest over the period

Note that we now use "present value" for the unpaid balance in the beginning of the operation. Thus, equation 1 above is written:

$$Int = CF_0 \times r$$

The future value is the initial capital (the unpaid balance or present value), plus the interests:

$$CF_1 = CF_0 + Int = CF_0 + CF_0 \times r = CF_0 \times (1 + r) \tag{2}$$

And the present value (or initial capital) is the future value minus the interests:

$$CF_0 = CF_1 - Int = CF_1/(1 + r) \tag{3}$$

Example 4

You borrowed money over one year at a rate of 4.1% per year. In the end of the year, you pay back a total of 2,914.8 to the lender, interests included. What was the amount you borrowed?

Note that this question is the same as "what is the present value of a future cash flow of 2,914.8 to be received in the end of year 1 if the discount rate is 4.1% per year?"

[3] We assume here that the interests are *postpaid* (paid in the end of the period) as it is by far the most common case. See prepaid interest for the other one.

Applying equation 3 above, we get:

$$CF_0 = 2,914.8/(1 + 4.1/100) = 2,914.8/1.041 = 2,800$$

What happens if there is more than one period, as often in the real life, where we borrow or save over a few years? The basic principles stay the same (how to remove the cost of financing to get the present value, the way to calculate the interest on one period only), but we have to separate two cases, depending on the way interests are calculated and earned. These cases are known as *simple interests method* and *compound interests method*.

3 Simple interests method

3.1 Simple interests

In the simple interests case, the interests produced or due on a given sum (capital) are never added to the initial capital: they are never *capitalized*. Thus, if no movement affects the initial capital, the interests stay the same period after period: they are calculated on the same amount. The interest over n period is simply n times the interest of one period.

Note that the simple interests method is usually used for short period of times. As a rule of thumb, you might assume that for all operations which are shorter than one year at origin, we use simple interest calculations.

3.2 Future value calculation

As we used CF_1 to note a cash flow to be paid or received in the end of period 1 (i.e. the future value at time 1), we will note CF_n a cash flow to be received or paid in the end of period n, that is, the future value at time n.

From the definition of simple interest above, we know that the interests over n period of time is n times the interest over one period. Thus we get:

$$CF_n = CF_0 + n \times Int = CF_0 + n \times CF_0 \times r = CF_0 \times (1 + nr) \tag{4}$$

Example 5

A deposit on a savings account will yield a 0.25% interest year *per month*. What is the amount available on the account after 8 months if the initial deposit was 3,500? Assume that no other movement affect the account over the 8 months.

Note that this question is the same as "what is the future value of a 3,500 deposit at a periodic rate of 0.25% after 8 periods if we use the simple interests method?"

We get:

$$CF_8 = 3,500 \times (1 + 8 \times 0.25/100) = 3,570$$

3.3 Present value calculation

Getting the present value from the future value is straightforward with equation 4 above, and we get:

$$CF_0 = CF_n/(1 + nr) \tag{5}$$

Example 6

What is the present value of a 12,646.32 cash flow to be received in 5 days from now, if the discount rate is 0.01% per day?

$$CF_0 = 12,646.32/(1 + 5 \times 0.01/100) = 12,640$$

3.4 Prepaid interest

So far in the loan examples we used, interest was paid by the borrower to the lender in the end of each period. It sometimes happens that interest is paid in the beginning of the period instead – though it is calculated in the same way. This convention is known as prepaid interest.

The table below summarizes the differences between the cash flows in the prepaid and postpaid interest methods in a single period case. L is the loan amount and r the interest rate for one period. The cash flows are noted CF_0 and CF_1 and are shown from the borrower point of view: if negative, it is a payment to the lender.

Cash flow	CF_0	CF_1	$CF_1 - CF_0$
Postpaid interest	L	$-L \times (1 + r)$	$-L \times r$
Prepaid interest	$L \times (1 - r)$	$-L$	$-L \times r$

What does it change? We can note that the amount of interests paid in each case is the same: $L \times r$, that is, the difference between time 1 and time 0 cash flows. But in the prepaid case, the interests are paid in advance, leading to a smaller amount available to the borrower – who would pay the same amount of interest though: the prepaid case is more expensive, because the interests paid are the same, but the capital actually lent is smaller.

This means that we cannot compare the rates on prepaid and postpaid operations directly: for the same nominal rate, the prepaid operation is more expensive, as the rate actually paid is

$$\frac{L \times r}{L \times (1 - r)} = \frac{r}{1 - r} > r$$

Example 7

Assume a 1,000 loan over one month at a 0.4% monthly rate, prepaid interests.

The interests on the loan are $1,000 \times 0.4/100 = 4$.

As interests are prepaid, the borrower receives $1,000 - 4 = 996$ at time 0, and pays back $1,000$ in total at time 1. The interest rate really paid over the operation is thus:

$$4/996 = 4.016\%$$

Finally note that we introduce prepaid interest in the global section of simple interests calculation, because the use cases of prepaid interests are usually on short term operations, where simple interest calculation is assumed.

4 Compound interests method

4.1 Compound interests: interest on interest

In the compound interests method, interests are *capitalized* (added to the capital) in the end of each period, thus starting to produce additional interest: this is known as *interest on interest*.

The compound interest method is generally used for all financial operations which initial duration is more than one year. For example, it is typically used for all medium to long term loans, for investment decision techniques, etc.

The origin of the compound interest method is best illustrated with an example.

Example 8

Imagine you borrow 2,000 over 2 years, at a 4.5% interest rate *per year*. You make an agreement with the lender by which you will repay for the loan in the end of year 2: there is no capital repayment in the end of the first year. Still, as the base period is one year, you owe interests in the end of the first year: $2,000 \times 4.5/100 = 90$. Thus, the cash flows on the loan would be as follows: you receive 2,000 at time 0, you pay 90 at time 1 (interests over the first period) and you pay 2090 at time 2 (capital repayment plus interests over the second period).

After discussion with the lender, you reach another agreement: you will not have to pay anything at time 1, in the end of the first period. Instead, you will pay everything at time 2, in the end of the second period. As is often said in finance, “there is no free lunch”: the interests you owe at time 1 over the first period are lent to you by the lender from time 1 to time 2, that is, over one period of time. Thus you have to pay interest on the first period interest: $90 \times 4.5/100 = 4.05$. Finally, with this new agreement, the cash flows are as follows: you receive 2,000 at time 0, nothing happens at time 1, and you pay 2,184.05 at time 2, that is, 2,000 for capital repayment, 2 times 90 for the interests over 2 periods of time, and 4.05 as interest on the interest of the first period.

4.2 Future value calculation

As interests are compounded (there is interest on interest in period 2, then interest on the interest on interest in period 3, etc.), the cash flows future values grow *geometrically*. Using the same notations as before, we get:

$$CF_n = CF_0 \times (1 + r)^n \tag{6}$$

Example 9

A 10,000 deposit is saved on an account paying 3.4% per year. No other movement affect the account in the next 5 years. What is the amount available in the end of year 5 if interests are compounded annually?

The amount available is simply the future value in year five, that is, CF_5:

$$CF_5 = 10,000 \times (1 + 3.4/100)^5 = 11,819.60$$

4.3 Present value calculation

From equation 6 above, the present value calculation is straightforward:

$$CF_0 = \frac{CF_n}{(1+r)^n} = CF_n \times (1+r)^{-n} \tag{7}$$

Example 10

How much should be deposited on a savings account paying 3.2% per year with yearly compounding to get 5,000 after 4 years (no other movements on the account)?

$$CF_0 = 5,000 \times (1 + 3.2/100)^{-4} = 4,408.10$$

4.4 Periodic rates

As you probably noticed, most of the times, interest rates are given on a yearly basis: it is not even necessary to specify “per year”, it is implicit.

But if the basis period of compounding is not a year but a month or any other duration, how to get the *periodic rate* (the rate for one period) from the yearly one?

The answer is not too difficult: the periodic rate can be deduced from the formulas we use for present and future value.

Example 11

A 100 deposit is made on a savings account. There are no other movements affecting the account and after one year, the total amount on the account is 104.8. What is the annual rate on the account?

This is straightforward, the rate is given by the relationship between the present and the future value, and there is only one period here:

$$\begin{aligned} 104.8 &= 100 \times (1 + r) \\ 1 + r &= 104.8/100 \\ r &= 104.8/100 - 1 \\ r &= 0.048 = 4.8\% \end{aligned}$$

So far, so good, the annual rate on the account is 4.8%. But what if we learn now that actually, the interests on this account were compounded monthly, that is, 12 times per year? What was the *monthly* rate?

Here, we use the relationship between the future and the present value, with compounding and 12 periods (months):

$$\begin{aligned} 104.8 &= 100 \times (1 + r)^{12} \\ (1 + r)^{12} &= 104.8/100 \\ (1 + r) &= (104.8/100)^{(1/12)} \\ r &= (104.8/100)^{(1/12)} - 1 \\ r &= 0.0039146 = 0.39\% \end{aligned}$$

We can easily generalize from the example above and find a relationship between the periodic and the annual rate in the case of interest compounding. We note r the annual rate as usual, and r_n the periodic rate for n periods per year (r_{12} would be the monthly rate, then):

$$(1 + r_n)^n = (1 + r)$$

$$r_n = (1 + r)^{(1/n)} - 1 \tag{8}$$

When the periodic rates are calculated that way, the annual rate is called an EAR for Effective Annual Rate. It is the annual rate that would give exactly the same future value with annual interest compounding – thus the adjective “effective”. It’s the “real” rate of the underlying operation.

For some reasons[4], the EAR are not always used, though. For the sake of simplicity, banks, for example, often use APR instead (even when interests are compounded). APR stands for Annual Percentage Rate, and cannot be simpler: the annual rate is the periodic rate times the number of periods per year. Conversely, the periodic rate is the APR divided by the number of periods per year.

In the following, whenever you are given a rate without any details, you may assume that it is an APR.

[4] Try to explain to your grandmother why we don’t simply divide by 12 to get the monthly rate.

4.5 Annuities and other series of cash flows

So far we used very simple operations in the examples: we only had a cash flow in the beginning (the present value), one in the end (the future value), and no intermediary cash flows. But in the real life, we often have a series of cash flows attached to a given operation: you repay for a loan in several installments over the years, a real estate company receives monthly rents from the tenants, an investor gets coupons and repayments on her bonds portfolio, etc.

In the most common case, when all cash flows are different and there is no special relationship between those (like growing at a constant rate), there is no choice but discount each cash flow in turn, to get a series of present values, and work on the series of present values to compare or add them for example.

Example 12

What is the total present value of the series of cash flows below if the discount rate is 8%?

Year	1	2	3	4
Cash flow	1,200	1,800	2,000	2,500

We have no choice but discount each cash flow in turn, using equation 7, and add the resulting 4 present values to get the total present value:

$$\frac{1,200}{(1+8/100)} + \frac{1,800}{(1+8/100)^2} + \frac{2,000}{(1+8/100)^3} + \frac{2,500}{(1+8/100)^4} = 6,079.56$$

Fig. 1 shows the detailed calculations on a spreadsheet. Notice the discounting formula in the data input area.

D5 =D4/(1+B1)^D3

	A	B	C	D	E	F
1	r	8.00%				
2						
3	Year	0	1	2	3	4
4	Cash flow		1,200.00	1,800.00	2,000.00	2,500.00
5	Discounted CF		1,111.11	1,543.21	1,587.66	1,837.57
6	Total present value	6,079.56				
7						

Figure 1: Discounting a series of cash flows

Luckily, there are shortcut formulas to get present or future values of series of constant (or increasing by a constant rate) cash flows, provided those are paid regularly over time (e.g. every quarter, or every year etc.). When the stream of cash flows is infinite, we speak of a perpetuity, when it has a finite duration and the cash flows are constant we call it a constant annuity.

Present value of a perpetuity

As we don't live forever, it seems to us that a perpetuity – an infinite stream of cash flows – is not something we often meet in the real life. Actually, perpetuities are quite common in finance: there are some perpetual bonds, and it is usually convenient to see the future stream of dividends paid on a stock as a perpetuity.

Of course, what we are interested in is the *present value* of a perpetuity, as, by definition, it does not have any future value.

The simplest case is a perpetuity of constant cash flows: we expect to get the same cash flow in the end of each period, forever. We note the constant cash flow *a* (for annuity, but again, the period might be days, weeks, months, etc.) and the present value is given by:

$$CF_0 = \frac{a}{r} \tag{9}$$

Example 13

What is the present value of a stream of cash flows paying 10,000 in the end of each year forever if the discount rate is 4.6%?

$$CF_0 = 10,000/(4.6/100) = 217,391.30$$

Some perpetuities are growing at a constant rate: every cash flow is calculated from the previous one, increased by a given percentage, noted *g* (for growth rate). Thus, if we write a_i for the cash flow paid in the end of year *i*, we have:

$$a_{i+1} = a_i \times (1 + g)$$

The present value of such a growing perpetuity is given by:

$$CF_0 = \frac{a_1}{r - g} \tag{10}$$

Example 14

A share of stock is supposed to pay a perpetual stream of dividends in the end of each year, growing by 0.8% per year. The first one (the expected dividend in the end of the current year) is 2.3. What is the present value of the stream if the discount rate is 12.4%?

$$CF_0 = \frac{2.3}{(12.4/100 - 0.8/100)} = 19.83$$

Present value of a constant but finite stream of cash flows

Even more common in finance than perpetuities are finite streams of constant cash flows: loan repaid over a given number of fixed monthly installments, for example. These are called *constant annuities*.

In this case, what we need is a formula to get the present value of the stream of cash flows: in the loan case, it would be the loan capital (the amount lent in the beginning), as it is actually the cash flow that is paid at time zero (from the lender to the borrower).

The formula can easily be deduced from the perpetuity one, once we realize that a finite stream of constant cash flows is the difference between two constant perpetuities, one starting now, and one starting right after the last payment. The complete proof does not belong to this introductory document and is left as an exercise to the reader.

Again, a is the constant cash flow, be it an annuity or other periodic payment:

$$CF_0 = a \times \frac{(1 - (1 + r)^{-n})}{r} \tag{11}$$

Getting the annuity from the present value is straightforward:

$$a = CF_0 \times \frac{r}{(1 - (1 + r)^{-n})} \tag{12}$$

Example 15

A 10 years loan will be repaid by ten annual installments of 13,076.54. The interest rate is 5.2%, what is the amount of the loan (the capital)?

Note that as it is a 10 years loan, we assume compound interests, and as payments are every year, the interests are compounded yearly.

The amount of the loan is the present value of the ten installments:

$$CF_0 = 13,076.54 \times \frac{(1 - (1 + 5.2/100)^{-10})}{(5.2/100)} = 100,000.01$$

Future value of a constant but finite stream of cash flows

Some operations, like savings, are a bit different: we deposit a given amount every period on an account paying a given rate, and we are interested in how much will be available in the end, that is, the future value of the annuity (or periodic payment).

Assume that n is the number of payments, and that we calculate the future value available right after the last payment. Adapt the calculation if the conditions are different: for example, add one period of interest if the future value is calculated one period after the last payment.

$$CF_n = a \times \frac{((1 + r)^n - 1)}{r} \tag{13}$$

Again, getting the annuity from the future value is trivial:

$$a = CF_n \times \frac{r}{((1 + r)^n - 1)} \tag{14}$$

Example 16

A family decides to save 500 every month on an account paying 0.35% interests per month (monthly compounding). How much will be available on the account after 12 years (right after the last payment)?

Note we assume that there are 12×12 payments here, as we have monthly payments.

$$CF_{144} = 500 \times \frac{((1 + 0.35/100)^{(12 \times 12)} - 1)}{(0.35/100)} = 93,410.49$$

Other periodic payments

For the sake of completeness, note that formulas exist for other kinds of periodic payments, like increasing ones (either by a fixed amount or by a fixed rate). These formulas are seldom used, though, and beyond the scope of this introductory document. If you need those, just search for “growing annuity” or something similar on the Internet.

Net present value

A very common operation is to get the *net present value* of a series of cash flows. In this case, you typically have to make an immediate payment to get the expected series of future cash flows. For example, you buy an apartment now and expect to get rents from the tenants in the future. The net present value is simply the present value of all the future cash flows, minus the payment you have to make immediately to get those. Note that the payment, being immediate, is already a present value, thus the comparison is legitimate. We thus get the present value, net of initial payment: the net present value. This concept is studied in depth in the capital budgeting and investment decision techniques.

Example 17

You are proposed to buy a voucher giving you the right to “ten years of holidays for free”. Actually, the voucher will allow you to spend 2,000 every year in various luxury resorts of the world, during ten years. The voucher price is 14,000 and the advertising insists on the “30% you save”. What is the net present value of this operation if your cost of opportunity (discount rate) is 7.8%?

As usual, because the basis period here is one year, we assume the interests are compounded annually.

The present value of the ten cash flows of 2,000 is given by equation 11:

$$2,000 \times \frac{(1 - (1 + 7.8/100)^{-10})}{(7.8/100)} = 13,542.07$$

Thus, the net present value of the proposal is:

$$13,542.07 - 14,000 = -457.93$$

It seems it is not such a good idea…

4.6 Other variables

As you probably noticed, equations 6 and 7 are actually the same one, we simply express it in a different way to find either the present value from the future value, number of years and rate, or find the future value from the present value, number of periods and rate.

Similarly, equations 11 and 12 are the same, the first version allows the calculation of the present value for a given annuity, and the second one the annuity for a given present value. And the same goes for future values of annuities.

It is sometimes possible to use these equations to find either the rate or the number of periods from the remaining variables. One common calculation is to find the rate (of return) from the present and future values of an investment with a given duration:

$$r = \left(\frac{CF_n}{CF_0}\right)^{\frac{1}{n}} - 1 \tag{15}$$

Example 18

You invest in a share of stock for 145.02. You do not receive any dividends or other payments, and you sell the stock 3 years later for 181.28. What is your return on this operation?

Here the present value is 145.02, the future value in year 3 is 181.28 and we are looking for the rate, we thus use equation 15 above:

$$r = \left(\frac{181.28}{145.02}\right)^{\frac{1}{3}} - 1 = 0.0772 = 7.72\%$$

Note that when we have a series of cash flows instead of simply one in the beginning (the present value) and one in the end (the future value), there is usually no explicit formula to find the rate r. This is because r would be the root of a polynomial expression, which degree is the number of periods: for n greater than 4, there is no general formula[5].

In finance, such a rate is called *internal rate of return* and the solution is found numerically. Fortunately, the numerical method is available in financial calculators and spreadsheets. Its name is usually `IRR` and its use is straightforward: just remember to give opposite signs to cash flows of opposite direction (cash flows you receive have a positive sign, cash flows you pay have a negative one, for example).

Example 19

Imagine that you invest in a share of stock for 780. You then receive a dividend in the end of each year, and the series of dividends is as follows:

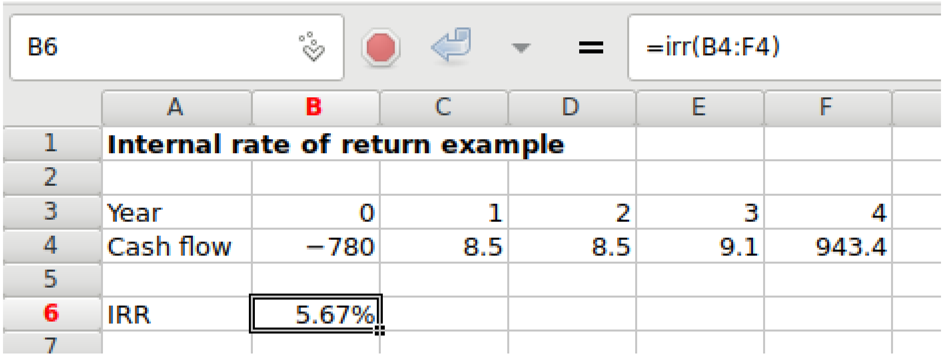
Year	1	2	3	4
Dividend	8.5	8.5	9.1	9.4

Finally, in year 4, right after receiving the dividend, you sell back the stock on the market for 934.

[5] This is known as the Abel-Ruffini theorem, see https://en.wikipedia.org/w/index.php?title=Abel%E2%80%93Ruffini_theorem&oldid=786729407

What is your (annual) return?

As we have intermediary cash flows (the dividends), there is no general formula to find the rate (which is the internal rate of return of your investment). Thus we use the spreadsheet instead. Fig. 2 shows the table we use: note that the year 0 cash flow (how much you paid for the stock) is negative, and that the cash flow in year 4 is the sum of the dividend payment and the sale of the stock. We then use the `IRR()` function on all the cash flows to find the rate. The answer is finally 5.67%.



B6 =irr(B4:F4)

	A	B	C	D	E	F
1	**Internal rate of return example**					
2						
3	Year	0	1	2	3	4
4	Cash flow	−780	8.5	8.5	9.1	943.4
5						
6	IRR	5.67%				
7						

Figure 2: Internal rate of return calculation

Finally, another common question is how many years are necessary to get a certain capital (future value), given the present value and the rate?

$$n = \frac{log(CF_n/CF_0)}{log(1+r)} \tag{16}$$

Example 20

How long does it take to double a capital which is saved at 7.5% per year?

As the capital is not given here, let us assume that CF_0=1. Thus, we should have CF_n=2. From equation 6 we can write:

$$2 = 1 \times (1 + 7.5/100)^n$$

This is equivalent to the expression in equation 16:

$$n = log(2/1)/log(1.075) = 9.58$$

Thus it would take a bit less than 10 years.

5 Loans repayment

Loans repayments are probably the most trivial use of the time value of money formulas. A loan is simply a stream of cash flows:

- the initial cash flow goes from the lender to the borrower and is the amount of the loan
- the others ones are from the borrower to the lender: they are called *installments* and should cover both the loan amount repayment and the interests payments.

The initial loan amount, often called the *capital*, might be repaid according to different schedules, depending on the agreement between the lender and the borrower. The three most common schedules are introduced below.

5.1 Loans repayment methods

The most common repayment methods are as follows:

Interest only loan this is the simplest one, by far: the initial amount is totally repaid on the last day of the loan, all the payments during the life of the loan are interest payments only (thus the name).

Constant repayments in this schedule, the same amount of capital is repaid in the end of each period, thus it is equal to the loan amount divided by the number of periods. The interest payment of the period should be added to this capital repayment, of course.

Constant annuity or installments the total installment (capital repayment plus interest payment) is the same in the end of each period and is thus calculated with the constant annuity formula, equation 12. Note that the balance between capital repayment and interests payment is different for each period.

Whatever the repayment method (or schedule) used, we always have:

- the sum of all capital repayments equals the initial capital (loan amount)
- the interest for a given period is always calculated on the unpaid balance in the beginning of this period, according to equation 1
- the end of period balance is the beginning of period balance, minus the capital repayment in the end of the period. Interests payments never ever influence the loan balance (it is the opposite actually)
- the beginning balance of a given period is the end balance of the previous one, for the first period, it is the loan amount.

To understand better the consequences of each schedule on the stream of cash flows, we will take an example in the section below.

5.2 Loan repayment table

The loan repayment table is the detailed table of the stream of cash flows related to a given loan, period after period.

The loan repayment table should provide at least[6] the following information for each period:

- period number, or better, exact date at which the payments related to the period are expected
- unpaid balance in the beginning of the period
- interests payment in the end of the period, based on the beginning of period unpaid balance and the periodic interest rate on the loan

[6] If an insurance is compulsory for the borrower, the periodic insurance premiums are often included in the loan repayment table.

- capital repaid in the end of the period
- total installment for that period, that is, sum of interest payment and capital repaid
- unpaid balance in the end of the period, to be reported in the beginning of the next period.

How to build a loan repayment table

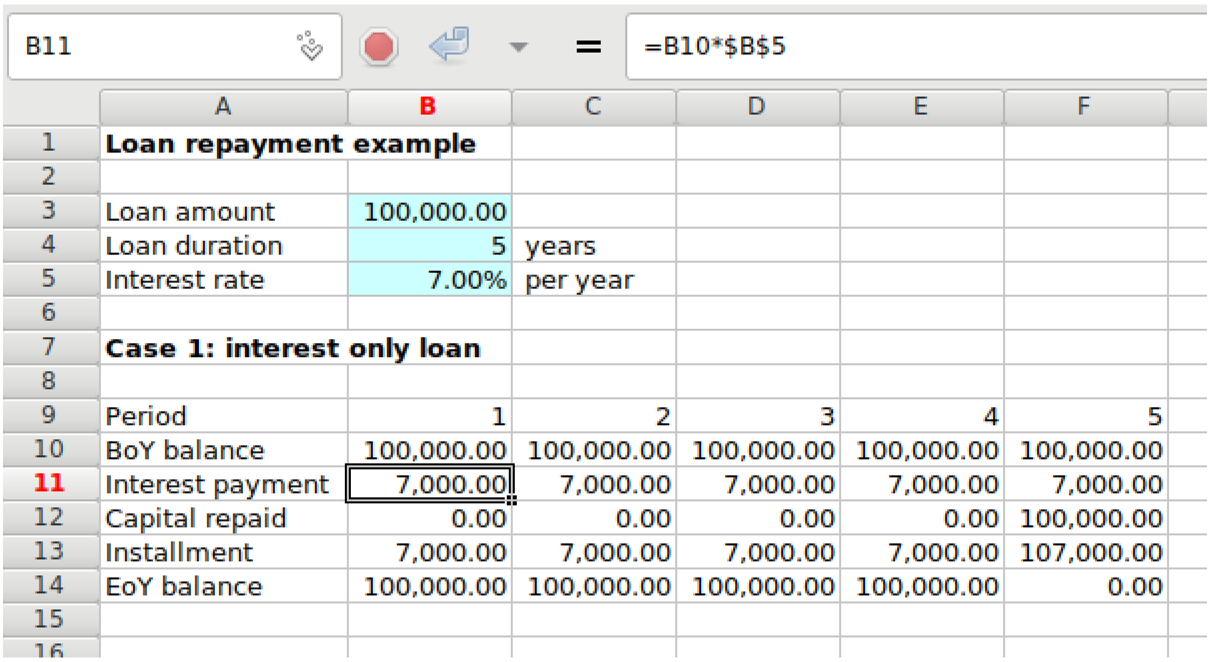
In order to better understand the contents and building procedure of such a table, let us take a loan as an example, and build the repayment table for each of the methods.

Example: Loan example

Consider a loan which initial amount is 100,000, to be repaid over 5 years with a 7% interest rate. Interests are compounded annually, as the year is the basic period.

The full example is available online[7] as a Google Spreadsheet[8].

Interest only loan


B11 =B10*B5

	A	B	C	D	E	F
1	**Loan repayment example**					
2						
3	Loan amount	100,000.00				
4	Loan duration	5	years			
5	Interest rate	7.00%	per year			
6						
7	**Case 1: interest only loan**					
8						
9	Period	1	2	3	4	5
10	BoY balance	100,000.00	100,000.00	100,000.00	100,000.00	100,000.00
11	Interest payment	7,000.00	7,000.00	7,000.00	7,000.00	7,000.00
12	Capital repaid	0.00	0.00	0.00	0.00	100,000.00
13	Installment	7,000.00	7,000.00	7,000.00	7,000.00	107,000.00
14	EoY balance	100,000.00	100,000.00	100,000.00	100,000.00	0.00
15						
16						

Figure 3: Interest only loan

We start with the “interest only loan” repayment method. Again, in this case, the capital repayment will happen in the end of the last period, by definition. The “capital repaid” row will thus always be empty, except on the last period column.

Of course, the borrower still has to pay interests on the unpaid balance, in the end of each period. As the capital is repaid once in the end, the unpaid balance never changes: it is the total amount of the loan until the last day (on which it is finally repaid). Thus, the periodic interests payments stay the same along the periods.

---

[7] Note that you cannot edit directly the spreadsheet online: you should download it and work on your local version.

[8] https://docs.google.com/spreadsheets/d/1Fb0lV7mBi3JoF_rR_4dUz-p1nJSIgM234QyIkaNDUpg/

Example: Loan example - Interest only loan case

In the loan example, the total amount, 100,000, will be repaid in the end of year five, which is the last period. Thus, the unpaid balance will stay the same and equal to the loan amount.

The interests will be paid at a 7% rate *every year* on this balance, thus the interest payment will always be:

$$100,000 \times 7/100 = 7,000$$

Fig. 3 shows the resulting table. Note the interest formula in B11, and the loan total repayment in F12. Finally, the unpaid balance in the end of year 5 is of course zero: the loan has been totally repaid.

Constant repayments

B26 = =B22-B24

	A	B	C	D	E	F
1	**Loan repayment example**					
2						
3	Loan amount	100,000.00				
4	Loan duration	5	years			
5	Interest rate	7.00%	per year			
16						
17	**Case 2: constant repayments**					
18						
19	Capital repaid	20,000.00				
20						
21	Period	1	2	3	4	5
22	BoY balance	100,000.00	80,000.00	60,000.00	40,000.00	20,000.00
23	Interest payment	7,000.00	5,600.00	4,200.00	2,800.00	1,400.00
24	Capital repaid	20,000.00	20,000.00	20,000.00	20,000.00	20,000.00
25	Installment	27,000.00	25,600.00	24,200.00	22,800.00	21,400.00
26	EoY balance	80,000.00	60,000.00	40,000.00	20,000.00	0.00
27						

Figure 4: Constant repayments

In this case, the loan amount (the capital) is repaid by equal repayments in the end of each period: each capital repayment is thus the loan amount divided by the number of periods.

The unpaid balance decreases accordingly, thus the interest payments decrease also (remember they are calculated on the unpaid balance). Finally the total installments are decreasing.

Example: Loan example - Constant repayments

The capital repaid in the end of each period is given by the loan amount divided by the number of periods:

$$100,000/5 = 20,000$$

> The rest follows and you can check the table on Fig. 4. Note the interests payments are indeed decreasing, as well as the total installments. Again, the unpaid balance in the end of year 5 is zero, as it should be.

Constant installments

B31 =B3*B5/(1-(1+B5)^-B4)

	A	B	C	D	E	F
1	**Loan repayment example**					
2						
3	Loan amount	100,000.00				
4	Loan duration	5	years			
5	Interest rate	7.00%	per year			
28						
29	**Case 3: constant annuities**					
30						
31	Constant annuity	24,389.07				
32						
33	Period	1	2	3	4	5
34	BoY balance	100,000.00	82,610.93	64,004.63	44,095.88	22,793.52
35	Interest payment	7,000.00	5,782.77	4,480.32	3,086.71	1,595.55
36	Capital repaid	17,389.07	18,606.30	19,908.75	21,302.36	22,793.52
37	Installment	24,389.07	24,389.07	24,389.07	24,389.07	24,389.07
38	EoY balance	82,610.93	64,004.63	44,095.88	22,793.52	0.00
39						

Figure 5: Constant installments

In the constant installments case, the sum of the interest payment and the capital repayment is constant over the periods. Thus, we have an initial cash flow, followed by constant and opposite direction cash flows over time: this is the constant cash flow or annuity case, as described in section 4.5 above.

More precisely, we should first calculate the constant installment according to equation 12. The rest of the calculations should be done period after period:

- calculate the interest payment on the period. For period one, interests are calculated on the loan amount, which is known. For the other periods, it is calculated from the beginning of period unpaid balance, which, by definition, is the end of the previous period unpaid balance
- calculate the capital repaid for the period by subtracting the interests just calculated from the constant installment
- calculate the end of period unpaid balance as usual by subtracting the capital repaid from the beginning of period unpaid balance
- go to next period and start again with the first point above.

Example: Loan example - Constant installments

First we calculate the constant installment (annuity) with equation 12:

$$a = 100,000 \times \frac{(7/100)}{(1-(1+7/100)^{-5})} = 24,389.07$$

As we calculated in the interest only loan case above, the interest payment on the first year is 7,000. Thus, the capital repaid in the first year should be:

$$24,389.07 - 7,000.00 = 17,389.07$$

From the capital repaid in the first year, we get the unpaid balance in the end of this year:

$$100,000.00 - 17,389.07 = 82,610.93$$

This will be the unpaid balance in the beginning of year 2, leading to the interest payment calculations etc.

Fig.5 shows the constant installments case repayment table. Notice the constant installments calculation in B31 with the formula in the input zone on the top of the figure. Notice also that, as it should, the unpaid balance in the end of the last period is zero: the loan has been repaid (the total of all capital repaid equals the amount of the loan).

Calculating the unpaid balance at any time

One common calculation on loans is the unpaid balance in the end of a given period. Of course, if the repayment table is available, it is easy to read the unpaid balance in it. But is there a direct way to know the unpaid balance in the end of a given period?

For the interest only loan case, the answer is immediate: unless we are in the end of the last period, the unpaid balance is the amount of the loan, as by definition it is totally repaid in the end of the last period.

For constant repayments, the answer is easy as well: the unpaid balance is given by the constant repayment, times the number of periods remaining.

Example 21

For the loan example above, in the constant repayment case, the constant repayment was $20,000$ per year, that is $100,000/5$. Thus, to look for the unpaid balance in, for example, the end of year 3, we simply calculate the number of years remaining, 5 - 3 = 2 and multiply this number by the constant repayment $20,000$. This yields an unpaid balance of $40,000$ in the end of year 3. You can check that this is indeed the case on Fig. 4.

Finally, for the constant installments case, it seems to be more difficult. But the trick is to remember that whatever the unpaid balance, it will be repaid by the remaining installments. It is thus the *present value of the remaining installments* which can be calculated by using equation 11 above.

Example 22

Still using the loan example in the previous section, we can calculate the unpaid balance in the end of year 3 in the constant installments case.

We remember that the constant installments was given by:

$$100,000 \times \frac{(7/100)}{(1-(1+7/100)^{-5})} = 24,389.07$$

In the end of year 3, there are 5 - 3 = 2 years remaining. The unpaid balance is thus the present value of the 2 installments remaining:

$$24,389.07 \times \frac{(1-(1+7/100)^{-2})}{(7/100)} = 44,095.88$$

Note that, because the constant installment was rounded to 2 decimal places, the above calculation might yield a result which is slightly different from the real one as read in the table. Let us be more precise by using the constant installment formula instead of its value in the calculation:

$$100,000 \times \frac{(7/100)}{(1-(1+7/100)^{-5})} \times \frac{(1-(1+7/100)^{-2})}{(7/100)}$$

We can simplify as $(7/100)$ is on both sides of the resulting calculation:

$$100,000 \times \frac{(1-(1+7/100)^{-2})}{(1-(1+7/100)^{-5})} = 44,095.88$$

You can check that the result is correct on Fig. 5.

Summary

- The origins of the time value of money are time preference and risk aversion.
- In the formulas, time preference is taken into account through the number of periods an operation would last.
- Risk aversion is taken into account through a risk premium which is included in the discount rate.
- The most common calculation is that of the *present value* of a cash flow or stream of cash flows: the present value is the value now, at present time, thus is suitable for a decision to be taken now.
- Two calculation systems exist: simple, and compound interests. The simple interest method is normally only used for short periods of time, under one year.
- Beyond the basic formulas, shortcuts exist for common cash flow patterns such as perpetuity or constant cash flows.
- The three most common methods for repaying loans are: interest only loans, constant repayments, and constant installments.
- A loan repayment table lists the cash flows related to a given loan, period after period. These cash flows include (but are not limited to) beginning of period balance, interest payment, capital repayment, total installment, end of period balance.

Exercises

Provide all answers with 2 decimal places, but round the final result only: remember you should never round any intermediary result.

1. What is the present value of a 48,000 cash flow to be received in 3 years from now if the discount rate is 5.8% APR and interests are monthly compounded?

2. You deposited 14,000 on a savings account 8 years ago, and totally forgot it. What is the available amount on the account now if interests are compounded every year and the account rate is 2.5%?

3. A share of stock will pay a 2.05 dividend in the end of the year. The dividend is paid every year and is expected to grow by 0.6% per year forever. If your opportunity rate is 11.6%, what is the maximum price you should pay for this stock?

4. A student borrows 24,000 from her bank at 4.7% (APR). The loan will be repaid by constant and monthly installments over 4 years.

   - What is the amount of the installment?
   - How much interest will be paid in the end of the first month?
   - What is the unpaid balance after 2 years?

5. You decide to prepare for a trip around the world in five years from now. You estimated the total cost of the trip to be 30,000. How much should you save every month on a savings account paying 2.3% per year (APR) to have the 30,000 available right before your trip? Assume 60 months of savings.

6. The unpaid balance on a 10 years loan is currently 76,396.74. The loan is repaid by constant annuities and its rate is 7.8%. What was the initial amount of the loan if there are 3 years remaining (i.e. we are in the end of year 7)?

7. Two retail banks are competing to attract the savings of customers. The first one promises “to double your capital over 10 years” when the second one simply announces a 7.5% rate. Assuming interests are yearly compounded, which one provides the best return?

Exercises answers

1. What is the present value of a 48,000 cash flow to be received in 3 years from now if the discount rate is 5.8% APR and interests are monthly compounded?

   Because interests are monthly compounded we need the monthly rate. As the rate provided is an APR, we simply need to divide it by 12. Of course, the number of periods is 3×12 months.

   Then from equation 7 we get:

   $$48,000 \times (1 + 5.8/(12 \times 100))^{-3\times12} = 40,351.16$$

2. You deposited 14,000 on a savings account 8 years ago, and totally forgot it. What is the available amount on the account now if interests are compounded every year and the account rate is 2.5%?

   From equation 6 we have:

   $$14,000 \times (1 + 2.5/100)^{8} = 17,057.64$$

3. A share of stock will pay a 2.05 dividend in the end of the year. The dividend is paid every year and is expected to grow by 0.6% per year forever. If your opportunity rate is 11.6%, what is the maximum price you should pay for this stock?

   Here we have a growing perpetuity: the cash flow (the dividend) grows by 0.6% per year forever. From equation 10 we get:

   $$2.05/(11.60/100 - 0.6/100) = 18.64$$

4. A student borrows 24,000 from her bank at 4.7% (APR). The loan will be repaid by constant and monthly installments over 4 years.

   As the interests are compounded monthly, we should use the APR divided by 12 as rate, and the periods are months.

   - What is the amount of the installment?
     The installment is a constant annuity, thus from equation 11 we get:

     $$24,000 \times \frac{((4.7/12)/100)}{(1 - (1 + (4.7/12)/100)^{-4\times12})} = 549.45$$

   - How much interest will be paid in the end of the first month?
     In the end of the first month, the beginning of period balance is the amount of the loan. The interests to pay are given by equation 1:

     $$24,000 \times ((4.7/12)/100) = 94.00$$

   - What is the unpaid balance after 2 years?
     After 2 years we still have 2 years (thus 48 months) remaining. From example 22 we have:

     $$24,000 \times \frac{(1 - (1 + (4.7/12)/100)^{-2\times12})}{(1 - (1 + (4.7/12)/100)^{-4\times12})} = 12,562.49$$

5. You decide to prepare for a trip around the world in five years from now. You estimated the total cost of the trip to be 30,000. How much should you save every month on a savings account paying 2.3% per year (APR) to have the 30,000 available right before your trip? Assume 60 months of savings.

   Here we want the *future value* of the monthly savings to be 30,000. Again, we have monthly compounding, thus the APR should be divided by 12 and the periods are months. From equation 14 we get:

$$30,000 \times \frac{((2.3/12)/100)}{((1+(2.3/12)/100)^{60}-1)} = 472.28$$

6. The unpaid balance on a 10 years loan is currently 76,396.74. The loan is repaid by constant annuities and its rate is 7.8%. What was the initial amount of the loan if there are 3 years remaining (i.e. we are in the end of year 7)?

   Here we know the unpaid balance and we are looking for the initial amount of the loan: we have to use the method of example 22, but reverse the fraction. We get:

$$76,396.74 \times \frac{(1-(1+7.8/100)^{-10})}{(1-(1+7.8/100)^{-3})} = 200,000.00$$

7. Two retail banks are competing to attract the savings of customers. The first one promises “to double your capital over 10 years” when the second one simply announces a 7.5% rate. Assuming interests are yearly compounded, which one provides the best return?

   To answer this question, we can either calculate the implicit rate in the first bank promise, or find by how much would a capital be multiplied after ten years if deposited in the second bank. We chose the first option: the rate is given by equation 15, and as the capital is supposed to double, we take 1 as present value, and 2 as future value:

$$2^{\frac{1}{10}} - 1 = 7.18\%$$

   Thus the second bank offers the best rate.